Unit 14 - Exponential and Logarithmic Functions

Day 1 - Real Exponents

<u>Objective</u> - You will simplify expressions + solve equations by using the properties of exponents.

Definition:	Example
If $n=1$, $b^n = b$	$13^1 = 13$
If $n>1$, $b^n = b \cdot b \cdot b \cdots b$ (n factors)	$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$
If $n=0$, $b^0 = 1$	$32^0 = 1$
If $b \neq 0$, $b^{-n} = \frac{1}{b^n}$	$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

Properties of Exponents

m and n are positive integers, a and b are real #'s.

Property	Definition
Product	$a^m \cdot a^n = a^{m+n}$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $b \neq 0$
Power of a Product	$(ab)^m = a^m b^m$
Quotient	$\frac{a^m}{a^n} = a^{m-n}$

<u>Defn of $b^{\frac{1}{n}}$</u>: For any real # $b \geq 0$ + any integer $n>1$

$b^{\frac{1}{n}} = \sqrt[n]{b}$ (Also true when $b<0$, n is odd)

$(-8)^{1/3} = \sqrt[3]{-8} = -2$

$9^{1/2} = \sqrt[2]{9} = 3$

<u>Rational Exponents</u>: For any nonzero #b, and any integers m and n with $n>1$ and m and n have no common factors:

$b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$ Except where $\sqrt[n]{b}$ is not a real #

on to Example ⟶

Examples

Examples	Answers
1) Evaluate: $\frac{2^4 \cdot 2^8}{2^5} = \frac{2^{12}}{2^5} = 2^7$	128
2) Evaluate: $\left(\frac{2}{5}\right)^{-2} = \frac{5^2}{2^2} = \frac{25}{4}$	$\frac{25}{4}$
3) Simplify: $(s^2 t^3)^5 = s^{10} t^{15}$	$s^{10} t^{15}$
4) Simplify: $\frac{x^3 y}{(x^4)^3} = \frac{x^3 y}{x^{12}} = x^{-9} y$	$\frac{y}{x^9}$
5) Simplify: $(81x^4)^{1/4} = 81^{1/4} x^{(4)1/4} = \sqrt[4]{81}\, x$	$= 3x$
6) Simplify: $\sqrt[6]{9x^3} \Rightarrow \sqrt[6]{3^2}\,\sqrt[6]{x^3} \longrightarrow$	$\sqrt[3]{3}\,\sqrt{x}$
7) Evaluate: $625^{3/4}$	125
8) Simplify: $\sqrt[3]{64 s^9 t^{15}}$	$45s^3 t^5$
9) Simplify: $\sqrt{r^7 s^{25} t^3}$	$r^3 s^{12} t \sqrt{rst}$
10) Solve: $86 = x^{4/3} + 5$	27

HW: Look in the paper/internet + find a house you think you would be able to afford on your salary after you have been working 1 year after college in your field.

14-2 Exponential Functions - Present Value of an Annuity (Mortgages)

<u>Objective</u> - You will find the present value of an annuity.

Present Value of an Annuity

$$P_n = P\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Annuity - Payments made at equal intervals.

P_n = Present Value of an annuity.
P = Periodic Payment (in dollars)
n = total # of payments
i = interest rate - $\frac{APR \times .01}{\text{\# of Payments/Yr.}}$

<u>Example</u>: You want to buy a house that costs \$195,000 and you have a down payment of \$20,000.

a) What would your monthly payments be if you get approved for a 30 year mortgage at 4.5% APR

$P_n = 195{,}000$ \$20,000 Down Payment

$i = \frac{4.5 \times .01}{12} = .00375$

$n = 12 \times 30 = 360$

Now $P_n = \$175{,}000$

$$P_n = P\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$175{,}000 = P\left[\frac{1-(1+.00375)^{-360}}{.00375}\right]$$

$$\frac{175{,}000}{197.361154} = \frac{197.361154\,P}{197.361154}$$

$$\boxed{P = 886.70}$$

b) How much money will be paid to the bank after 30 yrs

$886.70 \times 12 = 10{,}640.40$

$\times\ 30$

$\boxed{\$319{,}212}$

Example: You buy a new car for $27,000. Down Payment of $4000, 5 year loan.

Option #1

2.9% APR

#1

$P_n = 27{,}000$

$i = \frac{2.9 \times .01}{12} = .0024$

$n = 12 \times 5 = 60$

Now $P_n = \$23{,}000$

$$23{,}000 = P\left[\frac{1-(1+.0024)^{-60}}{.0024}\right]$$

$$23{,}000 = P(55.817855)$$

$$\boxed{P = \$412.05}$$

Option #2

$2000 Rebate
6.55% APR

#2 P_n Now = $21,000

$i = \frac{6.55 \times .01}{12} = .0055$

$n = 12 \times 5 = 60$

$$21{,}000 = P\left[\frac{1-(1+.0055)^{-60}}{.0055}\right]$$

$$21{,}000 = P(50.986533)$$

$$\boxed{P = \$411.87}$$

HW: 1) Calculate monthly payments for a 15 year mortgage on a $150,000 house. $5000 Down Payment, APR = 3.5%

2) 5-year loan to buy a $7000 pool. APR = 5.8%. Monthly Payments

Unit 14 - Day #3 Exponential Functions - (Con't)

Objective - You will find the future value of an annuity.

Future Value of an Annuity

$$F_n = P\left[\frac{(1+i)^n - 1}{i}\right]$$

Annuity - Payments made at equal intervals.

F_n = Future value of an annuity.

P = Periodic Payments (in dollars)

n = total # of payments.

i = interest rate - $\frac{APR \times .01}{\text{\# of Payments/year}}$

Example #1: You have just started your first "real" job and decide to start saving for retirement. If you invest \$100 every 2 weeks into an account averaging 4.5% APR, how much money will you have when you retire? (You choose the starting and ending ages).

22 years old → 65 years old. = 43 years

$i = \frac{4.5 \times .01}{26} = .00173$

$n = 43 \times 26 = 1,118$

$$F_n = P\left[\frac{(1+i)^n - 1}{i}\right]$$

$$F_n = 100\left[\frac{(1+.00173)^{1118} - 1}{.00173}\right]$$

$F_n = 3414.185349\,(100)$

$\boxed{F_n = \$341,418.54}$

2) You decide to start saving for a new car. The car you want to buy is \$19,000. If you are going to put money into an account for the next 4 years, how much money would you have to put into the account every <u>week</u> to have enough money? APR = 5%

$i = \frac{5 \times .01}{52} = 9.615\text{E-}4$ $n = 52 \times 4 = 208$

$F_n = \$19{,}000$

$19{,}000 = P\left[\frac{(1 + 9.615 \times 10^{-4})^{208} - 1}{9.615 \times 10^{-4}}\right]$ .0009615

$19{,}000 = P(230.1460181)$

$\boxed{P = \$82.56}$

Classwork / Homework   <u>Worksheet</u>

Day #4 - Compound Interest

$$A = P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow \text{Can't use } F_n = P\left[\frac{(1+i)^n - 1}{i}\right]$$

b/c F_n is for periodic payments.

A = Final Amount
P = Initial Investment
r = Annual Interest Rate
n = # of times compounded per year.
t = # of years.

Example: You invest $2000, compounded daily at 5% for 7 years. How much money will you have?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 2000\left(1+\frac{.05}{365}\right)^{365 \cdot 7}$$

$$\boxed{A = \$2{,}838.07} \rightarrow \text{so you earned } \$838.07$$

Example: You invest $1000 in an account that compounds interest monthly at 4% for 20 years. How much money will you have?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 1000\left(1+\frac{.04}{12}\right)^{12 \times 20}$$

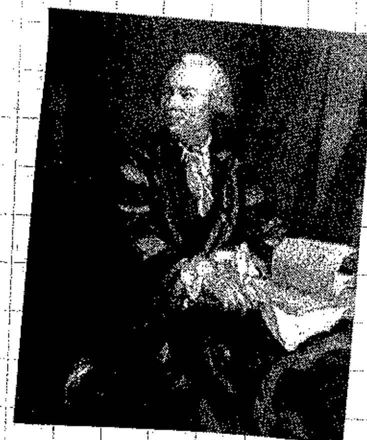
$$\boxed{A = \$2222.58}$$

Day #5 → The Number e

Objective – You will use the exponential function $y = e^x$

The Number e

- Discovered by Leonhard Euler (pronounced "Oiler")
- Deals with instantaneous compounding.



Formula used to compute interest:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Ex: Invest \$1.00 at 100% interest for 1 year.

How much money will you have if compounded:

Annually? $A = 1\left(1+\frac{1}{1}\right)^{1(1)} = \boxed{\$2.00}$

Semi-Annually? $A = 1\left(1+\frac{1}{2}\right)^{2(1)} = \boxed{\$2.25}$

Monthly? $A = 1\left(1+\frac{1}{12}\right)^{12(1)} = \boxed{\$2.61}$

Daily? $A = 1\left(1+\frac{1}{365}\right)^{365(1)} = \boxed{\$2.71}$

Instantaneously? $A = 1\left(1+\frac{1}{31,536,000}\right)^{31,536,000} = \boxed{\$2.72}$

Continuously Compounded Interest

$$A = Pe^{rt}$$

A = Final Amount
P = Initial Investment
r = Annual Interest Rate
t = # of years.

Example: You invest \$10,000 at 6.75% for 25 years. Determine how much money you will have if the interest is compounded

a) Semi-Annually:

$A = P\left(1+\frac{r}{n}\right)^{nt}$

$A = \$10{,}000\left(1+\frac{.0675}{2}\right)^{2(25)}$

$\boxed{A = \$52{,}574.62}$

b) Continuously

$A = Pe^{rt}$

$A = 10{,}000\, e^{(.0675)(25)}$

$\boxed{A = \$54{,}059.49}$

Example: As time passes, you forget things you have learned. The formula for this percentage of retained knowledge in t weeks is represented by $P = (100-a)e^{-bt} + a$ with a and b varying for each person. If 2 weeks have passed, find P if:

a) $a = 18 \quad b = 0.6$

$P = (100-a)e^{-bt} + a$

$P = (100-18)e^{-.6(2)} + 18$

$\boxed{P = 42.6979}$

b) $a = 16 \quad b = 0.7$

$P = (100-16)e^{-.7(2)} + 16$

$\boxed{P = 36.714144}$

HW: 2 Ex's